Empirical State Error Covariance Matrix for Batch Estimation*

Joe Frisbee

KBR Wyle Services, LLC

(NASA/JSC ISS Trajectory Operations and Planning)

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Abstract/Introduction

State estimation techniques effectively provide mean state estimates. However, the theoretical state error covariance matrices provided as part of these techniques often suffer from a lack of confidence in their abilities to describe the true uncertainty in the estimated states. By a reinterpretation of the equations involved in the weighted least squares algorithm, it is possible to directly arrive at an empirical state error covariance matrix. This proposed empirical state error covariance matrix will contain the effect of all error sources, known or unknown. Results are presented for a simple, two observer, measurement error only problem.

Background

- There are various estimation techniques:
  - Batch
  - Sequential (Kalman)
- Provide a theoretical state error covariance matrix describing estimate uncertainty under perfect process knowledge, maybe with process noise.
- Theoretical state error covariance matrices:
  - Do not include all error sources.
  - Too small but may be too large if improperly corrected.
  - Not trusted as a quantitative description of the state error.
- "Filter" error covariance matrices are, at best, qualitative estimates of the error. "Is the estimate good enough for..."
- There appears to be a lack of reference to any formal empirical state error covariance matrix for such estimation processes.

What about an empirical error covariance matrix?

- An empirical state error covariance matrix may be determined.
- This matrix will include <u>all</u> error sources, known or not!
- Why?
  - Actual observations contain true measurement errors.
  - Estimated measurements contain all other errors, known or not.
  - Therefore, measurement residuals contain all errors, known or not.
- What about unknown bias?
  - An empirical covariance matrix will not eliminate bias problems.
  - Biases will be part of the empirical uncertainty of the estimate.
  - Current theoretical covariance excludes bias altogether.

For Batch Estimation

- The empirical matrix is consistent with existing tools.
- The empirical matrix only requires the addition of side computations to any existing batch estimator.
- The empirical matrix is a straight forward statistic of a sample measurement process.
- The empirical matrix comes, as any statistic should, with a path to confidence intervals for elements of the matrix.

Typical Batch Estimation Presentation

- Minimize the total variance cost function of the weighted squares of the residuals:

$$J(x) = \frac{1}{2}(\mathbf{y} - \mathbf{H}x)^T \mathbf{W}(\mathbf{y} - \mathbf{H}x)$$

- Standard form of the solution:

$$\hat{x} = \left(\mathbf{H}^T \mathbf{W} \mathbf{H}\right)^{-1} \mathbf{H}^{\mathbf{T}} \mathbf{W} \mathbf{y}$$

- Identified error covariance matrix of the estimate:

$$P = \left(\mathbf{H}^{\mathrm{T}} \mathbf{W} \mathbf{H}\right)^{-1}$$

- This is usually where the story ends.

Empirical Error Covariance Matrix

- Rewrite the cost function using summation notation and in the form of a mean rather than a total variance:

$$J(x) = \frac{1}{2N}\sum_{i=1}^{N}(y_i - H_i x)^T W_i (y_i - H_i x)$$

- Follow the usual procedure, only keep the "N" and summation notation:

$$\hat{x} = \left(\frac{1}{N}\sum_{i=1}^{N} H_i^T W_i H_i\right)^{-1}\left(\frac{1}{N}\sum_{i=1}^{N} H_i^T W_i y_i\right)$$

Empirical Error Covariance Matrix

- Identify the coefficient term as the population error covariance matrix associated with the traditional batch estimation algorithm:

$$P_p = \left(\frac{1}{N} \sum_{i=1}^{N} H_i^{\,T} W_i H_i \right)^{-1}$$

- For the solution equation, combine all of the terms inside the summation and identify the summation argument, $\tilde{x}_i$, as an effective state vector measurement residual. (Note: the i-th effective state vector residual is not the generalized inverse solution to the i-th measurement residual expression.)

$$\hat{x} = \frac{1}{N} \sum_{i=1}^{N} \left(A_i y_i \right) = \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_i$$

Empirical Error Covariance Matrix

- From the last equation, it follows directly that the empirical (sample) population error covariance matrix is given by:

$$\tilde{P}_p = \frac{1}{N-1}\sum_{i=1}^{N}\left[(\tilde{x}_i - \hat{x})(\tilde{x}_i - \hat{x})^T\right]$$

- After the usual iterative process for batch estimation, the estimated correction is essentially zero and the above may be approximated by:

$$\tilde{P}_p = \frac{1}{N}\sum_{i=1}^{N}\left(\tilde{x}_i \tilde{x}_i^{\,T}\right)$$

Empirical Error Covariance Matrix

- From statistics, the covariance of a sample mean is just the sample covariance of the population divided by the number of samples:

$$\tilde{P} = \frac{1}{N^2} \sum_{i=1}^{N} \left(\tilde{x}_i \tilde{x}_i^T \right)$$

- This is the empirical error covariance matrix for the batch estimation process written in terms of effective state measurement residuals.
- What does this expression look like using the original terms in the state update equation?

Empirical Error Covariance Matrix

- Using some of the original expressions, the empirical error covariance matrix may be written as:

$$\tilde{P} = \frac{1}{N^2} P_p \left[\sum_{i=1}^{N} \left(H_i^T W_i y_i y_i^T W_i H_i \right) \right] P_p$$

- Using the relationship between the population covariance and the traditional state error covariance, the empirical form of the batch estimate error covariance matrix may be written as:

$$\tilde{P} = P \left[\sum_{i=1}^{N} \left(H_i^T W_i y_i y_i^T W_i H_i \right) \right] P$$

Empirical Error Covariance Matrix

- This last equation may be written in appended matrix form as:

$$\tilde{P} = P\mathbf{H}^{\mathrm{T}}\mathbf{WYWH}\,P$$

- Note: the center matrix, $\mathbf{Y}$, <u>is not</u> the outer product of the traditional appended residual vector with itself. $\mathbf{Y}$ is an appended block diagonal matrix with each block being the outer product of an individual measurement residual vector with itself:

$$\mathbf{Y} = \begin{bmatrix} y_1 y_1^T & 0 & \cdots & 0 \\ 0 & y_2 y_2^T & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & y_n y_n^T \end{bmatrix}$$

Empirical Error Covariance Matrix

Points to note

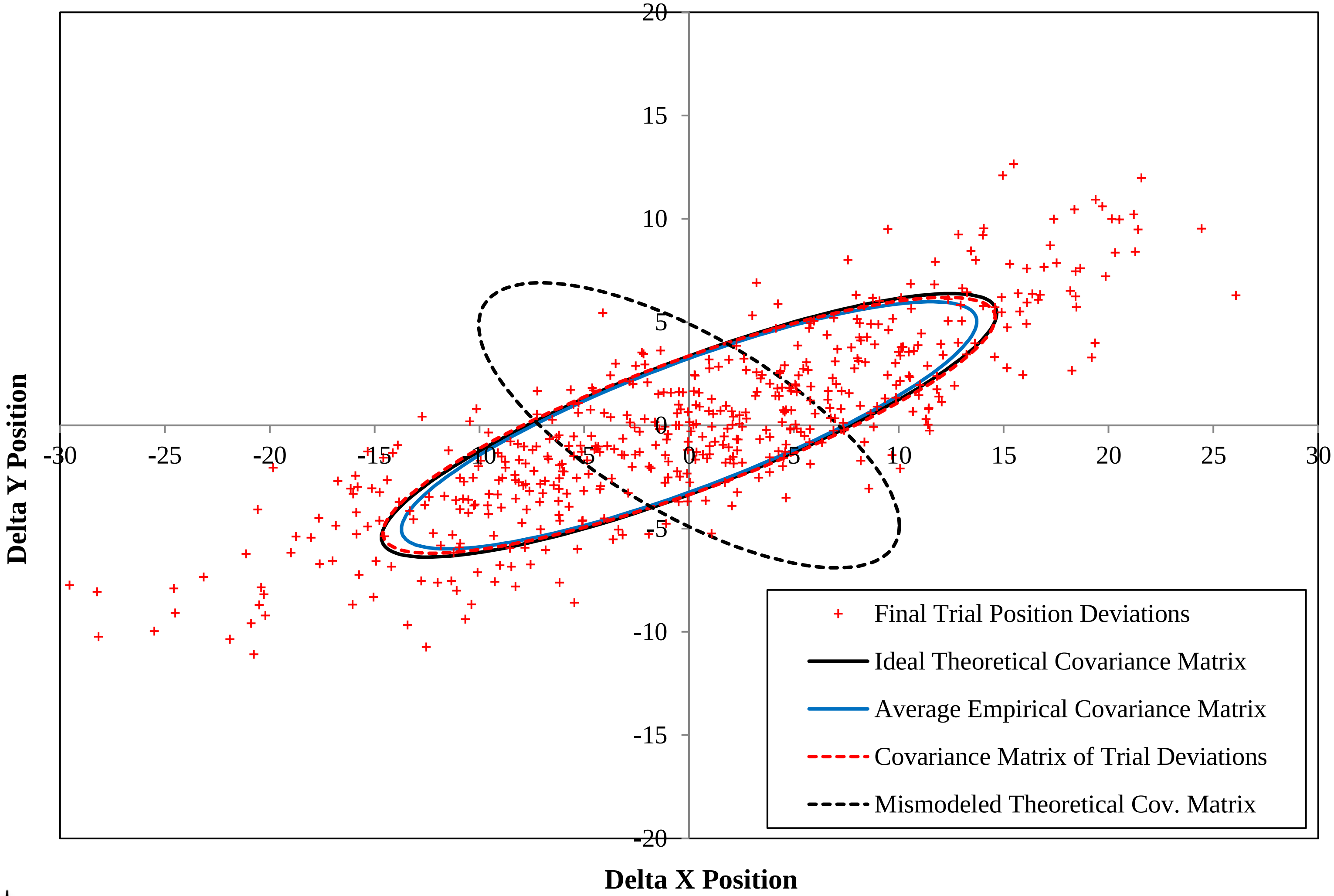
- All information necessary for computation of an empirical state error covariance matrix is present within any existing batch estimate process.
- Assuming that the measurement residual weighting matrix is the inverse of the measurement error covariance, the expected value of the empirical error covariance matrix is the same matrix as is usually computed for the traditional batch estimate:

$$\boldsymbol{E}\left[\tilde{P}\right] = P\left[\sum_{i=1}^{n}\left(H_i^T W_i \boldsymbol{E}\left[y_i y_i^T\right] W_i H_i\right)\right] P = P$$

Example: Stationary Triangulation

- Two range only observers ranging a target object.
- Sample range measurements were generated using given but different sensor noise values for each of the two sensors.
- Standard batch process subject to faulty range measurement weights: the sensor uncertainties were swapped prior to data processing.
- Data generated:
  - Theoretical, 2 sigma, error ellipse using correct weights.
  - A field of multiple predicted locations generated by repeating the experiment numerous times w/ calculated 2 sigma covariance matrix.
  - Empirical, 2 sigma, error ellipse from the measurement residuals under the influence of the erroneous measurement weights.
  - Theoretical, 2 sigma, error ellipse using swapped weights.

Example: Stationary Triangulation



Error Covariance Confidence Interval

- Sample processes, by their very nature, provide statistically uncertain results for each parameter determined through the sampling process.
- A standard part of most statistical analyses is to determine the uncertainty associated with an estimated parameter.
- The batch estimation procedure provides a theoretical estimate of the uncertainty in the estimated state and, as just shown, an empirical estimate of the state uncertainty.
- It is also possible to directly consider the uncertainty in the estimates of the error covariance matrices themselves.

Error Covariance Confidence Interval

- Recalling the earlier form of the empirical error covariance matrix:

$$\tilde{P} = P\left[\sum_{i=1}^{n}\left(H_i^T W_i y_i y_i^T W_i H_i\right)\right]P$$

- If the weighting matrix is the inverse of the expected variance of the measurement, then each residual maybe rewritten in terms of a normalized vector and the square root matrix of the expected variance corresponding to each measurement's error.

$$y_i = W_i^{"-\frac{1}{2}"} u_i = S_i u_i$$

Error Covariance Confidence Interval

- After substitution, bringing the traditional matrix inside the sum and simplification, the empirical error covariance matrix is of the form:

$$\tilde{\mathrm{P}} = \sum_{\mathrm{i}=1}^{\mathrm{n}} \left(\mathrm{B}_{\mathrm{i}} \mathrm{u}_{\mathrm{i}} \mathrm{u}_{\mathrm{i}}^{\mathrm{T}} \mathrm{B}_{\mathrm{i}}^{\mathrm{T}} \right)$$

- Let $b_{i:k}$ represent any explicit row, k, of B_i above. It can then be shown that the contribution of the i-th observation to the row m and column n element of the prior expected value of the error covariance matrix is:

$$\boldsymbol{E}\left[\tilde{P}_{i:mn}\right] = b_{i:m} b_{i:n}^{T} = P_{i:mn}$$

Error Covariance Confidence Interval

- It can also be shown that the expected value of the contribution of each observation to the variance of each component of the prior expected value of the error covariance matrix is:

$$\boldsymbol{V}\left[\tilde{P}_{i:mn}\right] = \left(b_{i:m} b_{i:m}^T\right)\left(b_{i:n} b_{i:n}^T\right) + \left(b_{i:m} b_{i:n}^T\right)^2$$

- With the final results:

$$\boldsymbol{E}\left[\tilde{P}_{mn}\right] = \sum_{i=1}^{N} P_{i:mn} = P_{mn}$$

$$\boldsymbol{V}\left[\tilde{P}_{mn}\right] = \sum_{i=1}^{N} \boldsymbol{V}\left[\tilde{P}_{i:mn}\right] = \boldsymbol{V}\left[P_{mn}\right]$$

Error Covariance Confidence Interval

- Considering the diagonal terms, it is possible to write each of the previous results in terms of a gamma distributed variable with "to be determined" shape, K, and scale, θ, parameters:

$$\mathrm{P}_{mm} = \boldsymbol{E}\left[\Gamma\left(\mathrm{K}_{mm}, \theta_{mm}\right)\right] = \mathrm{K}_{mm}\theta_{mm}$$

$$\boldsymbol{V}\left[\mathrm{P}_{mm}\right] = \boldsymbol{V}\left[\Gamma\left(\mathrm{K}_{mm}, \theta_{mm}\right)\right] = \mathrm{K}_{mm}\theta_{mm}^{2}$$

- It is possible to solve for both shape and scale parameters. These two parameters define the distribution of the uncertainty of the diagonal covariance matrix element with which they are associated under the prior expectations of the measurement errors.

Error Covariance Confidence Interval

<u>What about an off-diagonal matrix element</u>?

- If the magnitude of the expected value is much greater than the square root of the variance of that element then the previous method will work.
- If the expected value is approximately zero, then the off-diagonal element will have, approximately, a Gaussian distribution.
- If neither of the above is the case then the associated correlation coefficient of the element in question should be investigated. There is already an approximate, known distribution associated with the sample correlation coefficient. (This may also work for the two previous conditions as well.)

Empirical Error Covariance Matrix

Other Comments

- All of the confidence interval discussion revolves around theoretical expectations. Thus the confidence intervals formally describe the theoretical uncertainty in the theoretical state error covariance matrix.
- It is possible to determine “empirical” confidence intervals and this should result in “K-factor” like modifications to the variances of the individual covariance elements.
- Future work and applications…

Summary

- It is possible to directly and unambiguously determine a formally correct empirical state error covariance matrix which describes the batch filter estimate state vector uncertainty.
- It is directly possible to determine theoretical confidence intervals associated with the elements of the batch state error covariance matrix. These intervals apply specifically to the traditional batch filter error covariance matrix and, though not presented, it is possible to determine confidence intervals specifically for the empirical error covariance matrix elements.
- All of this within one estimate without any knowledge of the true state nor having to perform any systematic comparisons.